Motion Compensation for Ship Borne Radars and Lidars

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ABSTRACT. Three radars and a lidar were sip borne during the RICO experiment. Their data requires correction of the measured Doppler velocities for motion of the instruments as well as determination of the spatial position above the Earth to which each datum pertains. This report gives a thorough analysis of how to use the data to calculate the required quantities for the RICO experiment. The analysis is general enough to be apply to all ship and aircraft borne sensors.

1. INTRODUCTION

The NOAA/K cloud and precipitation scanning radar, the University (Univ.) of Miami X-band and W-band cloud radars, and the NOAA Mini-MOPA scanning Doppler lidar were aboard the Research Vessel (R/V) Seward Johnson during the Rain in Cumulus over the Ocean (RICO) experiment. The RICO data from those instruments were obtained from northwest of the island of Barbuda during January 2005. The trade-wind swell caused substantial motion of the R/V Seward Johnson. Data from all of those instruments require correction for the ship's motion.

In Section 2, the ships' coordinate system is defined as is the angular rate of rotation of the ship and the transformation of coordinates between the ship's and the Earth's coordinate systems. The most essential equation for motion compensation is the relationship between the velocities at any two points on the ship; that relationship is given in Section 2. Section 3 shows how to calculate the radars' and lidar's radial direction in the Earth's coordinate system and how to calculate the antenna's radial velocity for correction of the Doppler velocity. Each datum from radars and the lidar must be associated with the spatial position from which the back scattering occurred. Section 4 shows how to calculate those spatial positions in latitude, longitude, and height. The lidar consists of many optical elements that are in motion relative to one another as viewed from an inertial reference frame. Section 5 derives, in detail, the motion correction for the lidar's measured Doppler frequency shift and velocity. In Section 5, the relationship of the measured Doppler frequency to atmospheric and lidar motions is derived, and the correction of the atmospheric velocity measured by the lidar is given. Section 7 compares the correction for motions using the ship's versus the lidar's motion detection systems which allows the two systems to be intercompared. Such intercomparison can determine the accuracy of the systems. Section 7 shows how either motion detection system can be used to correct data from any of the radars or the lidar. The survey of the positions of the instruments aboard the ship is essential to motion compensation. The surveyed positions are given in the Appendix.

Motion correction equations in this report were derived at sea without reference to published literature (Section 5 on the correction of Doppler velocity measured by the lidar was written after return from the voyage). Comparison with previous publications (e.g., Edson *et al.*, 1998; Schulz *et al.*, 2005) shows variations of the formulation that arise because of different definitions of coordinate systems, angular rates, and Euler angles. Much confusion could result unless such definitions are clearly stated. A particular distinction is made by Edson *et al.* (1998) between gyro-stabilized systems and strapped-down systems. The lidar's motion compensation system is the strapped-down type. The POS MV system aboard the R/V Seward Johnson is the strapped-down type. The description of the data recorded by the POS MV system is given in Corcoran and Prank (2003). The angular rates of body motion form a vector. For the RICO experiment, the components of the angular rate vector are given by the lidar's system in the lidar's coordinate system and by the ship's POS MV system in the ship's coordinate system. That is unlike formulations stated by Edson *et al.*, 1998 and Schulz *et al.*, 2005; those formulations use components of the angular rate vector in Earth-fixed coordinates.

2. THE SHIP'S COORDINATE SYSTEM AND MOTION DATA

A vector is a quantity that is independent of coordinate systems, and, as such, it is denoted without a superscript. Arbitrary vector $\mathbf{U}$ is an example. The same vector $\mathbf{U}$ with its components obtained on the ship's, or Earth's, or lidar's coordinate system is denoted by $\mathbf{U}^S$, or $\mathbf{U}^E$, or $\mathbf{U}^L$, respectively. Note that vectors are denoted by bold type. Axes 'forward', 'starboard' (i.e., 'right of forward'), and 'downward' constitute, in that order, an orthogonal right-handed coordinate system. It is called the ship's coordinate system. Unit vectors aligned along the axes in the positive sense are denoted by

$$\begin{array}{rl} \text{forward} & \widehat{\mathbf{x}} \\ \text{starboard} & \widehat{\mathbf{y}} \\ \text{downward} & \widehat{\mathbf{z}} \end{array}$$

The caret denotes that a vector has unit magnitude.

The angle about direction forward is 'roll' ϕ, about direction starboard is 'pitch' θ, about direction downward is 'heading' ψ. These angles are known as Euler angles. The angular rates are $\frac{d\phi}{dt}$, $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, which have units of radians per second. An angular rate vector can be formed by the ordered triple

$$\mathbf{\Omega}^S = \begin{pmatrix} \widehat{\mathbf{x}} \cdot \mathbf{\Omega} \\ \widehat{\mathbf{y}} \cdot \mathbf{\Omega} \\ \widehat{\mathbf{z}} \cdot \mathbf{\Omega} \end{pmatrix} = \begin{pmatrix} \Omega_x^S \\ \Omega_y^S \\ \Omega_z^S \end{pmatrix} = \begin{pmatrix} \frac{d\phi}{dt} \\ \frac{d\theta}{dt} \\ \frac{d\psi}{dt} \end{pmatrix}. \tag{1}$$

The superscript S denotes that the components of any vector are obtained in the ship's coordinate system. The dot product in (1) using the ships' unit vectors $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}})$ produces the components of the vector in the ship's coordinate system.

Consider a position vector $\mathbf{r}^S(AB)$ from point A on the ship to point B on the ship, where the components of $\mathbf{r}^S$ are in the ship's coordinate system. Given that the ship can be treated as a rigid body, the velocity of point B relative to point A is the cross product $\mathbf{\Omega}^S \times \mathbf{r}^S(AB)$. The cross product is defined by

$$\mathbf{\Omega} \times \mathbf{r} = \begin{pmatrix} \widehat{\mathbf{x}} \cdot \mathbf{\Omega} \times \mathbf{r} \\ \widehat{\mathbf{y}} \cdot \mathbf{\Omega} \times \mathbf{r} \\ \widehat{\mathbf{z}} \cdot \mathbf{\Omega} \times \mathbf{r} \end{pmatrix} = \begin{pmatrix} \Omega_y r_z - \Omega_z r_y \\ \Omega_z r_x - \Omega_x r_z \\ \Omega_x r_y - \Omega_y r_x \end{pmatrix}, \tag{2}$$

wherein the absence of a superscript denotes that (2) is true independent of the coordinate system. Note that the argument (AB) of the vector $\mathbf{r}^S(AB)$ is omitted in (2) for clarity. The cross product (2) is always performed in the ship's coordinate system because $\mathbf{r}^S(AB)$ is a constant in that coordinate system and $\mathbf{\Omega}^S$ is measured in that coordinate system.

The Earth's coordinate system is north, east, down in that order; it is a right-handed coordinate system. The motion detection systems report velocities relative to the solid Earth with the components of that velocity in the Earth's coordinate system. Denote such velocities with superscript E, e.g. $\mathbf{V}^E$. Let the velocity of point A on the ship be denoted by $\mathbf{V}^E(A)$ and the velocity at point B on the ship be denoted by $\mathbf{V}^E(B)$. Given that the ship is a rigid body, the relationship between those two velocities is

$$\mathbf{V}^E(A) = \mathbf{V}^E(B) + \left(\mathbf{\Omega}^S \times \mathbf{r}^S(AB)\right)^E \tag{3}$$

This relationship between velocities is central to motion compensation of radar and lidar data and can also be used to correct data from the tower-mounted sonic anemometer. As the notation in (3) implies, the vector $\mathbf{\Omega}^S \times \mathbf{r}^S(AB)$ is determined in the ship's coordinate system and must have its components determined in the Earth's coordinate system before it can be added within (3). How to calculate $\left(\mathbf{\Omega}^S \times \mathbf{r}^S(AB)\right)^E$ is the topic of the next two subsections.

2.1 The Euler-Angle Rotation Matrix

The coordinate transformation from the Earth's coordinate system to that of the ship is needed. The transformation matrix is obtained from the product of three rotation matrixes. Transformation matrices are denoted by bold type. Begin with a Cartesian coordinate system aligned with the Earth's coordinate system (north, east, down). Rotate that coordinate system until it coincides with the ship's coordinate system as follows. First, rotate that coordinate system about its heading axis (i.e., down axis) by ψ; denote the rotation matrix by $\mathbf{C}$.

$$\mathbf{C} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second rotate about the new pitch axis by θ; denote the rotation matrix by $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Last, rotate about the new roll axis by ϕ; denote the rotation matrix by $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

The coordinate transformation matrix is the product $\mathbf{ABC}$ as follows:

$$\mathbf{Q} \equiv \mathbf{ABC} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi & \cos\theta\cos\phi \end{bmatrix}$$

The inverse of this matrix is its transpose because it is an orthogonal transformation.

$$\mathbf{Q}^{-1} = \mathbf{Q}^{T} = \begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

At each time step, new values of the angles ϕ, θ, ψ are determined by integration of the angular rate vector $\mathbf{\Omega}$. At each time step the above matrix $\mathbf{Q}$ is computed from the ϕ, θ, ψ. The definitions above apply to both the lidar's and the ship's coordinate systems.

2.2 How to Determine $\mathbf{\Omega} \times \mathbf{r}$ in the Earth's Coordinate System

The above definition of the Euler angles ϕ, θ, ψ gives the transformation between components of a vector in the ship's coordinate system to the Earth's coordinate system. Let $\mathbf{U}^E$ denote any vector when its components are in the Earth's coordinate system, and let $\mathbf{U}^S$ be that same vector in the ship's coordinate system. Then,

$$\mathbf{U}^S = \mathbf{Q}\mathbf{U}^E$$
$$\mathbf{U}^E = \mathbf{Q}^{-1}\mathbf{U}^S$$

For use in (3) the required computation at each time step is to calculate the components of $\mathbf{\Omega}^S \times \mathbf{r}^S$, then transform that vector to the Earth's coordinate system at which point it is denoted by $(\mathbf{\Omega} \times \mathbf{r})^E$. Since (2) is $\mathbf{\Omega}^S \times \mathbf{r}^S = (\mathbf{\Omega} \times \mathbf{r})^S$is in the ship's coordinate system we obtain $(\mathbf{\Omega} \times \mathbf{r})^E$ from

$$(\mathbf{\Omega} \times \mathbf{r})^E = \mathbf{Q}^{-1} (\mathbf{\Omega} \times \mathbf{r})^S \tag{4}$$

The above also applies to the lidar's coordinate system for which case superscript S is replaced by L.

3. HOW TO DETERMINE THE RADAR'S RADIAL DIRECTION IN THE EARTH'S COORDINATE SYSTEM AND CALCULATE THE ANTENNA'S RADIAL VELOCITY

First, define the radar radial unit vector in the ship's coordinate system. Assume that the radar's measurement of azimuth φ is level with the main deck, and zero degrees azimuth is forward, and azimuth is positive if the rotation is from forward toward starboard. Assume that the, radar's measurement of elevation ε is positive from the plane containing the main deck. Then the unit vector pointing outward from the radar's antenna in the ship's coordinate system is

$$\widehat{\mathbf{p}}^S = \begin{pmatrix} \widehat{\mathbf{x}} \cdot \widehat{\mathbf{p}}^S \\ \widehat{\mathbf{y}} \cdot \widehat{\mathbf{p}}^S \\ \widehat{\mathbf{z}} \cdot \widehat{\mathbf{p}}^S \end{pmatrix} = \begin{pmatrix} p_x^S \\ p_y^S \\ p_z^S \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\varepsilon \\ \sin\varphi\cos\varepsilon \\ -\sin\varepsilon \end{pmatrix} \tag{5}$$

Verify that it is a unit vector:

$$\widehat{\mathbf{p}}^S \cdot \widehat{\mathbf{p}}^S = \left(\cos^2\varphi + \sin^2\varphi\right)\cos^2\varepsilon + \sin^2\varepsilon = \cos^2\varepsilon + \sin^2\varepsilon = 1$$

Similar to (4), the radar radial unit vector in the Earth's coordinate system is

$$\widehat{\mathbf{p}}^E = \mathbf{Q}^{-1}\widehat{\mathbf{p}}^S \tag{6}$$

Assume the convention that motion toward the radar antenna is negative and motion away from the radar antenna is positive. The radial velocity correction in Earth coordinates is

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E$$

where $\mathbf{v}^E$ is the velocity at the phase center of the radar's antenna; of course, $\mathbf{v}^E$ is in the Earth's coordinate system. This correction must be added to (not subtracted from) the radar's measurement of radial velocity.

It is a good approximation that the Univ. Miami radars are pointed straight up relative to the main deck. Therefore,

$$\widehat{\mathbf{p}}^S = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

The ship's heave is the velocity component perpendicular to the main deck; heave is positive for downward motion. The correction to the Univ. Miami radar's Doppler velocity is the negative of the ship's local heave at the location of the radar antenna, namely,

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E = \left(\mathbf{Q}^{-1}\widehat{\mathbf{p}}^S\right) \cdot \mathbf{v}^E =$$

$$\begin{aligned} &(-\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi)\, v^E_{north} \\ &+ (\cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi)\, v^E_{east} \\ &+ (-\cos\theta\cos\phi)\, v^E_{down} \end{aligned}$$

Here, we used the fact that the Earth's coordinate system is north, east, down in that order. Also used was the following matrix multiplication:

$$\mathbf{Q}^{-1}\widehat{\mathbf{p}}^S =$$

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi \\ \cos\psi\sin\phi - \cos\phi\sin\theta\sin\psi \\ -\cos\theta\cos\phi \end{pmatrix}$$

More generally, for (5) we have that

$$\widehat{\mathbf{p}}^E = \mathbf{Q}^{-1}\widehat{\mathbf{p}}^S =$$

$$\begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\varepsilon \\ \sin\varphi\cos\varepsilon \\ -\sin\varepsilon \end{pmatrix} =$$

$$\begin{pmatrix} \cos\theta\cos\varepsilon\cos\psi\cos\varphi - (\sin\varepsilon)(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) + (\cos\varepsilon\sin\varphi)(-\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi) \\ \cos\theta\cos\varepsilon\cos\varphi\sin\psi + (\cos\varepsilon\sin\varphi)(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi) - (\sin\varepsilon)(-\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi) \\ -\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi \end{pmatrix} \tag{7}$$

Therefore,

$$\widehat{\mathbf{p}}^E \cdot \mathbf{v}^E =$$

$$\begin{aligned} &[\cos\theta\cos\varepsilon\cos\psi\cos\varphi - (\sin\varepsilon)(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi) + (\cos\varepsilon\sin\varphi)(-\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi)]\, v^E_{north} \\ &+ [\cos\theta\cos\varepsilon\cos\varphi\sin\psi + (\cos\varepsilon\sin\varphi)(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi) - (\sin\varepsilon)(-\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi)]\, v^E_{east} \\ &+ [-\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi]\, v^E_{down} \end{aligned} \tag{8}$$

This latter expression is needed to correct the Doppler velocity measured by the NOAA/K radar. The required data for (8) are the angles φ, ε, ϕ, θ, ψ, and velocity $\mathbf{v}^E$; that data was recorded with the NOAA/K radar data.

There exists software to produce images of NOAA/K radar's reflectivity, Doppler velocity, and other measured parameters as functions of the radar's elevation angle. Those images are distorted by ship motion. However, if the elevation above the Earth's horizon is calculated and used, then the the images will be corrected. Similar to (5)

$$\widehat{\mathbf{p}}^E = \begin{pmatrix} \widehat{\mathbf{x}} \cdot \widehat{\mathbf{p}}^E \\ \widehat{\mathbf{y}} \cdot \widehat{\mathbf{p}}^E \\ \widehat{\mathbf{z}} \cdot \widehat{\mathbf{p}}^E \end{pmatrix} = \begin{pmatrix} p_x^E \\ p_y^E \\ p_z^E \end{pmatrix} = \begin{pmatrix} \cos\varphi^E \cos\varepsilon^E \\ \sin\varphi^E \cos\varepsilon^E \\ -\sin\varepsilon^E \end{pmatrix} \tag{9}$$

Recall that the both the ship's and the Earth's coordinate systems have their axes positive in the downward direction. Therefore, ε^E is elevation angle that is positive upward from the horizon. Also, φ^E is azimuth in radians eastward from north. From (7), the value of p_z^E is already known to be

$$p_z^E = -\cos\theta\cos\phi\sin\varepsilon - \sin\theta\cos\varepsilon\cos\varphi + \cos\theta\cos\varepsilon\sin\phi\sin\varphi.$$

Consequently, from (9) the desired elevation angle in radians for display of radar images is

$$\varepsilon^E = \arcsin\left(-p_z^E\right) \tag{10}$$

Note that ε^E depends on radar azimuth φ and elevation ε and on the ship's pitch θ and roll ϕ, but not on the ship's heading ψ. Unlike scans obtained with the radar on solid ground, the azimuth φ^E varies with the ship motion.

4. HOW TO CALCULATE THE SPATIAL POSITION OF EACH DATUM OF RADAR AND LIDAR MEASUREMENT

The radius of the Earth is $R_{Earth} = 6378$ km. Note that the GPS geoid is about 35 m above sea level in the RICO study area; that corresponds to altitude $z = -35$ m in ship's coordinates. The NOAA/K radar and the lidar measure each datum within their averaging volume at their recorded range r_{ange}, elevation ε, and azimuth φ. The Univ. Miami radars record only range because their elevation is fixed at perpendicular to the main deck. The position vector of each datum relative to the antenna in the Earth's coordinate system is the product $\left(\widehat{\mathbf{p}}^E\right)(r_{ange})$. See (6) above for the calculation of $\widehat{\mathbf{p}}^E$. The ship's POS MV system gives latitude, longitude and altitude of the center of the radars' antennas and of the lidar's accelerometer box as functions of time: $(lat\,(t)\,, lon\,(t)\,, z\,(t))$. The unit of $lat\,(t)$ and $lon\,(t)$ is decimal degrees, and the unit of $z\,(t)$ is meters. Increments of latitude and longitude are calculated in the small angle approximation, e.g., $\sin\left(r_{ange}/R_{Earth}\right) \simeq r_{ange}/R_{Earth}$. The increment of latitude in degrees associated with the north component of $\left(\widehat{\mathbf{p}}^E\right) r_{ange}$ is

$$\Delta lat = \frac{180}{\pi R_{Earth}} \left(\widehat{p}_{north}^E\right) r_{ange}$$

The increment of longitude associated with the east component of $\left(\widehat{\mathbf{p}}^E\right) r_{ange}$ is

$$\Delta lon = \frac{180}{\pi R_{Earth} \cos\left(lat\right)} \left(\widehat{p}_{east}^E\right) r_{ange}$$

The increment of altitude is

$$\Delta z = \left(\widehat{p}_{down}^E\right) r_{ange}$$

Finally, the latitude, longitude, and altitude of each datum of radars and lidar is

$$lat_{datum} = lat\,(t) + \Delta lat \tag{11}$$

$$lon_{datum} = lon\,(t) + \Delta lon \tag{12}$$

$$z_{datum} = z\,(t) + \Delta z \tag{13}$$

The height H above the sea surface for each datum can be obtained to within about 0.2 m. In Appendix A the following is given: The main deck is 1.2 m above the sea surface adjacent to the reference point 'ref' marked by the **X** drilled into the deck, and the decks rise by about 0.64 m to the flux tower at the bow. The NOAA/K radar and Univ. Miami radars are close enough to point 'ref' to use the 1.2 m height above sea level of the main deck.

The lidar is close enough to the bow so as to estimate its height above sea level as an additional 0.6 m. Using the surveyed heights of the radars and lidar given in Appendix A, we have

$$H_{lidar} = 1.8 + 4.84 - (\Delta z)_{lidar} \tag{14}$$

$$H_{NOAA/K} = 1.2 + 5.30 - (\Delta z)_{NOAA/K} \tag{15}$$

$$H_{W-band} = 1.2 + 2.88 - (\Delta z)_{W-band} \tag{16}$$

These heights are given in meters and are positive for positions above the sea surface; Δz is subtracted above because Δz is negative for positions above the sea surface.

5. DERIVATION OF THE CORRECTION OF DOPPLER VELOCITY MEASURED BY THE LIDAR

The lidar consists of very many optical elements on an optical table as well as in a periscope. The accelerations of the ship cause the lidar to be in a non-inertial coordinate system such that the various optical elements are in motion relative to one another. The simplification is that the optical table is a rigid body such that the relative velocities of optical elements mounted on it are constrained by rigid body equations. Likewise, the periscope is a rigid body constrained relative to the optical table because both are mounted to the sea container. The motion of the lidar necessitates a correction to the measured Doppler velocity in order to determine the atmosphere's velocity in the Earth's coordinate system. What velocities of which optical elements must be used to determine the correction? The derivation below uses the rigid body constraints to demonstrate that it is only the velocity of the scanning mirror that determines the correction. The scanning mirror is the last mirror before the beam exits to the atmosphere. In the derivation below, it suffices to use several mirrors at arbitrary positions rather than to analyze the specific positions of mirrors within the lidar.

5.1 Doppler Shift Resulting from Reflection from a Mirror

Consider the Doppler shift resulting from reflection from a mirror. Let $\mathbf{k}_i$ be the wave vector of light incident on the mirror and let ω_i be its frequency. Let $\mathbf{k}_r$ be the wave vector of light reflected from the mirror and let ω_r be its frequency. Choose a point on the mirror; at time t; its position is $\mathbf{x}(t)$. The incident and reflected waves have the same phase at $\mathbf{x}(t)$. For example, a crest of the incident wave causes a crest of the reflected wave, etc. Therefore, at time t

$$\mathbf{k}_r \cdot \mathbf{x}(t) - \omega_r t = \mathbf{k}_i \cdot \mathbf{x}(t) - \omega_i t$$

Similarly at a time $t + \Delta t$:

$$\mathbf{k}_r \cdot \mathbf{x}(t+\Delta t) - \omega_r (t+\Delta t) = \mathbf{k}_i \cdot \mathbf{x}(t+\Delta t) - \omega_i (t+\Delta t)$$

The difference of these equations is

$$\mathbf{k}_r \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)] - \omega_r [(t+\Delta t) - t] = \mathbf{k}_i \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)] - \omega_i [(t+\Delta t) - t]$$
$$\mathbf{k}_r \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)]/\Delta t - \omega_r = \mathbf{k}_i \cdot [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)]/\Delta t - \omega_i$$

The velocity of the mirror at point $\mathbf{x}(t)$ is, in the limit of small Δt,

$$\mathbf{u}(t) = [\mathbf{x}(t+\Delta t) - \mathbf{x}(t)]/\Delta t$$

Hence, the Doppler shift imposed by reflection from the moving mirror is

$$\omega_i - \omega_r = (\mathbf{k}_i - \mathbf{k}_r) \cdot \mathbf{u}(t) \tag{17}$$

5.2 Doppler Shift Resulting from a Sequence of Mirrors Constrained to Move as a Rigid Body

The mirrors within the lidar's periscope move as the periscope rotates and as the scanning mirror within the periscope moves to direct the outward beam. The other mirrors of the lidar system are constrained by rigid body motion of the sea container. Consider the inertial reference frame that coincides with the position of the beam splitter at time t, and has the same velocity as the beam splitter at that time t. Light from the beam splitter goes both to the mixer and toward the atmosphere. The considered inertial reference frame is convenient because we need not consider changes of the light's frequency between where the light is created and the beam splitter.

Consider the light propagating from the beam splitter to mirror 1, then to mirror 2, then to mirror 3, as the light propagates toward the atmosphere. The position vectors from the beam splitter to mirrors 1, 2 and 3 are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively. The position vector from mirror 1 to mirror 2 is $\mathbf{r}_{12}$, and from mirror 2 to mirror 3 is $\mathbf{r}_{23}$. Relative to the beam splitter at time t, mirrors1, 2 and 3 have velocities $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, respectively. The relationship between the position vectors is

$$\begin{aligned}
&\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}_{12} \\
&\mathbf{r}_3 = \mathbf{r}_2 + \mathbf{r}_{23} = \mathbf{r}_1 + \mathbf{r}_{12} + \mathbf{r}_{23}
\end{aligned} \tag{18}$$

The wave vector of the light propagating from the beam splitter to mirror 1 is $\mathbf{k}_1$ and that light has frequency ω_1. The wave vector of the light propagating from mirror 1 to mirror 2 is $\mathbf{k}_2$ and that light has frequency ω_2. The wave vector of the light propagating from mirror 2 to mirror 3 is $\mathbf{k}_3$ and that light has frequency ω_3. Finally, the light reflected from mirror 3 has wave vector $\mathbf{k}_4$ and frequency ω_4. Repeated application of (17) gives

$$\begin{aligned}
\omega_1 - \omega_2 &= (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{u}_1 \\
\omega_2 - \omega_3 &= (\mathbf{k}_2 - \mathbf{k}_3) \cdot \mathbf{u}_2 \\
\omega_3 - \omega_4 &= (\mathbf{k}_3 - \mathbf{k}_4) \cdot \mathbf{u}_3
\end{aligned} \tag{19}$$

Adding these equations gives the frequency difference between the light at the beam splitter and the output light after reflection from mirror 3 as follows

$$\omega_1 - \omega_4 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{u}_1 + (\mathbf{k}_2 - \mathbf{k}_3) \cdot \mathbf{u}_2 + (\mathbf{k}_3 - \mathbf{k}_4) \cdot \mathbf{u}_3 \tag{20}$$

To excellent approximation in the limit $|\mathbf{u}_1|/c \ll 1$, where c is the speed of light, wave vector $\mathbf{k}_1$ is parallel to separation vector $\mathbf{r}_1$ such that it can be written as $\mathbf{k}_1 = a\mathbf{r}_1$, where a is a constant at any given time. Similarly, $\mathbf{k}_2 = b\mathbf{r}_{12}$, $\mathbf{k}_3 = c\mathbf{r}_{23}$. Using (18) we have

$$\begin{aligned}
\mathbf{k}_1 &= a\mathbf{r}_1 \\
\mathbf{k}_2 &= b\mathbf{r}_{12} = b\,(\mathbf{r}_2 - \mathbf{r}_1) \\
\mathbf{k}_3 &= c\mathbf{r}_{23} = c\,(\mathbf{r}_3 - \mathbf{r}_2)
\end{aligned}$$

The rigid body constraint is that the velocity of any given point $\mathbf{r}$ on the rigid body relative to the velocity of the beam splitter is

$$\mathbf{u} = \boldsymbol{\Omega} \times \mathbf{r} \tag{21}$$

where $\boldsymbol{\Omega}$ is the angular rotation-rate vector of the rigid body at time t, and $\mathbf{r}$ is the position vector from the beam splitter to the given point. Because the lidar's sea container and the ship move together as a rigid body, $\boldsymbol{\Omega}$ in (21) and below is the same angular rate that is denoted by $\boldsymbol{\Omega}$ in previous sections. Given any 3 vectors $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$, the triple scalar product obeys $\mathbf{U} \cdot \mathbf{V} \times \mathbf{W} = \mathbf{V} \cdot \mathbf{W} \times \mathbf{U}$. Therefore,

$$\begin{aligned}
(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{u}_1 &= (\mathbf{k}_1 - \mathbf{k}_2) \cdot \boldsymbol{\Omega} \times \mathbf{r}_1 \\
&= \boldsymbol{\Omega} \cdot \mathbf{r}_1 \times (\mathbf{k}_1 - \mathbf{k}_2) \\
&= \boldsymbol{\Omega} \cdot \mathbf{r}_1 \times (a\mathbf{r}_1 - b\,(\mathbf{r}_2 - \mathbf{r}_1)) \\
&= (a + b)\,\boldsymbol{\Omega} \cdot \mathbf{r}_1 \times \mathbf{r}_1 - b\boldsymbol{\Omega} \cdot \mathbf{r}_1 \times \mathbf{r}_2 \\
&= -b\boldsymbol{\Omega} \cdot \mathbf{r}_1 \times \mathbf{r}_2
\end{aligned}$$

$$\begin{aligned}
(\mathbf{k}_2 - \mathbf{k}_3) \cdot \mathbf{u}_2 &= (\mathbf{k}_2 - \mathbf{k}_3) \cdot \boldsymbol{\Omega} \times \mathbf{r}_2 \\
&= \boldsymbol{\Omega} \cdot \mathbf{r}_2 \times (\mathbf{k}_2 - \mathbf{k}_3) \\
&= \boldsymbol{\Omega} \cdot \mathbf{r}_2 \times (b\,(\mathbf{r}_2 - \mathbf{r}_1) - c\,(\mathbf{r}_3 - \mathbf{r}_2)) \\
&= \boldsymbol{\Omega} \cdot \mathbf{r}_2 \times (-b\mathbf{r}_1 - c\mathbf{r}_3) \\
&= -b\boldsymbol{\Omega} \cdot \mathbf{r}_2 \times \mathbf{r}_1 - c\boldsymbol{\Omega} \cdot \mathbf{r}_2 \times \mathbf{r}_3
\end{aligned}$$

$$\begin{aligned}(\mathbf{k}_3-\mathbf{k}_4)\cdot\mathbf{u}_3 &= \mathbf{k}_3\cdot\mathbf{\Omega}\times\mathbf{r}_3-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3\\ &= \mathbf{\Omega}\cdot\mathbf{r}_3\times\mathbf{k}_3-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3\\ &= \mathbf{\Omega}\cdot\mathbf{r}_3\times(c\,(\mathbf{r}_3-\mathbf{r}_2))-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3\\ &= -c\mathbf{\Omega}\cdot\mathbf{r}_3\times\mathbf{r}_2-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3\end{aligned}$$

Substitution of the above into (20) gives

$$\omega_1-\omega_4=-b\mathbf{\Omega}\cdot\mathbf{r}_1\times\mathbf{r}_2-b\mathbf{\Omega}\cdot\mathbf{r}_2\times\mathbf{r}_1-c\mathbf{\Omega}\cdot\mathbf{r}_2\times\mathbf{r}_3-c\mathbf{\Omega}\cdot\mathbf{r}_3\times\mathbf{r}_2-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3$$

Since $\mathbf{r}_1\times\mathbf{r}_2=-\mathbf{r}_2\times\mathbf{r}_1$ and $\mathbf{r}_2\times\mathbf{r}_3=-\mathbf{r}_3\times\mathbf{r}_2$, we have

$$\omega_1-\omega_4=-\mathbf{k}_4\cdot\mathbf{\Omega}\times\mathbf{r}_3 \tag{22}$$

The magnitude of $\mathbf{k}_4$ is ω_4/c. The rotation rate $\mathbf{\Omega}$ is measured by the lidar's motion compensation system (as well as by the ship), and $\mathbf{r}_3$ can be measured and expressed in the lidar's motion compensation coordinate system (as well as in the ship's coordinate system).

At this point the lidar beam can be considered as outgoing from the optical table and propagating to the periscope. The Doppler shift of the outgoing beam as caused by motion of the lidar's sea container is given by (22). According to (22), that Doppler shift contains no effect from mirrors other than the final mirror.

5.3 Doppler Shift Resulting from a Sequence of Mirrors within the Lidar's Periscope

For the purpose of directing the laser beam, the lidar's periscope rotates about the vertical direction relative to the lidar's sea container. The final laser beam orientation before exit into the atmosphere is produced by rotation of the scanning mirror within the periscope. At time t let the angular velocity of the periscope, relative to the sea container, be denoted by $\mathbf{\Omega}^{\text{periscope}}$. Let $\mathbf{k}_3$ be the wave vector of the light incident on the center of rotation of the first mirror within the periscope, and let its frequency be ω_3. That is, the first mirror in the periscope is at position $\mathbf{r}_3$ relative to the beam splitter and is the final mirror 3 in the above section. The center of rotation of mirror 3 is subject to the rigid body constraint in the above section. The wave vector of the light propagating from that first mirror within the periscope to the next mirror within the periscope is $\mathbf{k}_4$ and that light has frequency ω_4. The wave vector of the light propagating to the center of rotation of the scanning mirror is $\mathbf{k}_5$ and that light has frequency ω_5. Finally, the light reflected from the center of rotation of the scanning mirror has wave vector $\mathbf{k}_O$ and frequency ω_O. Thus, $\mathbf{k}_O$ and ω_O describe the laser beam that exits into the atmosphere (hence the subscript O is mnemonic for 'output'). There exists a Doppler spread caused by motion of different positions on the scanning mirror and on the first mirror within the periscope relative to their center of rotation; in the following, that Doppler spread is neglected.

Continuing the sequence of frequency differences (19) we have

$$\omega_4-\omega_5=(\mathbf{k}_4-\mathbf{k}_5)\cdot\mathbf{u}_4 \tag{23}$$

$$\omega_5-\omega_O=(\mathbf{k}_5-\mathbf{k}_O)\cdot\mathbf{u}_5 \tag{24}$$

However, the velocities $\mathbf{u}_4$ and $\mathbf{u}_5$ obey a more general formula than (21). The rigid body constraint is that the velocity of any point within the periscope, relative to the velocity of the beam splitter, is

$$\mathbf{u}=\mathbf{\Omega}\times\mathbf{r}+\mathbf{\Omega}^{\text{periscope}}\times\mathbf{r}^{\text{periscope}} \tag{25}$$

where $\mathbf{r}^{\text{periscope}}$ is the position vector from the center of rotation of the first mirror in the periscope to any point in the periscope, and $\mathbf{r}$ is the position vector from the beam splitter to the same point in the periscope. If the periscope is not rotating then $\mathbf{\Omega}^{\text{periscope}}=0$ and (25) becomes the rigid body constraint (21) because the sea container and periscope are one rigid body if $\mathbf{\Omega}^{\text{periscope}}=0$. If the sea container is not rotating, then $\mathbf{\Omega}\times\mathbf{r}=0$ and (25) gives a circular motion about mirror 3. Substituting (23-24) in (25) gives

$$\begin{aligned}\omega_4-\omega_5 &= (\mathbf{k}_4-\mathbf{k}_5)\cdot\mathbf{u}_4\\ &= (\mathbf{k}_4-\mathbf{k}_5)\cdot\mathbf{\Omega}\times\mathbf{r}_4+(\mathbf{k}_4-\mathbf{k}_5)\cdot\mathbf{\Omega}^{\text{periscope}}\times\mathbf{r}_{34}^{\text{periscope}}\\ \omega_5-\omega_O &= (\mathbf{k}_5-\mathbf{k}_O)\cdot\mathbf{u}_5\\ &= (\mathbf{k}_5-\mathbf{k}_O)\cdot\mathbf{\Omega}\times\mathbf{r}_5+(\mathbf{k}_5-\mathbf{k}_O)\cdot\mathbf{\Omega}^{\text{periscope}}\times\mathbf{r}_{35}^{\text{periscope}}\end{aligned} \tag{26}$$

Similar to the derivation in the previous section, we define constants d and e, and we approximate the wave vectors as being parallel to the separation vectors between the mirrors as follows:

$$\begin{aligned}\mathbf{k}_4 &= d\mathbf{r}_{34}^{\text{periscope}} = d\left(\mathbf{r}_4 - \mathbf{r}_3\right) \\ \mathbf{k}_5 &= e\mathbf{r}_{45}^{\text{periscope}} = e\mathbf{r}_{35}^{\text{periscope}} - e\mathbf{r}_{34}^{\text{periscope}} = e\left(\mathbf{r}_5 - \mathbf{r}_4\right)\end{aligned} \tag{27}$$

Then,

$$\begin{aligned}\left(\mathbf{k}_4 - \mathbf{k}_5\right)\cdot\mathbf{\Omega}\times\mathbf{r}_4 &= \mathbf{\Omega}\cdot\mathbf{r}_4\times\left(\mathbf{k}_4 - \mathbf{k}_5\right) \\ &= \mathbf{\Omega}\cdot\mathbf{r}_4\times\left(d\left(\mathbf{r}_4 - \mathbf{r}_3\right) - e\left(\mathbf{r}_5 - \mathbf{r}_4\right)\right) \\ &= \mathbf{\Omega}\cdot\mathbf{r}_4\times\left(-d\mathbf{r}_3 - e\mathbf{r}_5\right) \\ &= -d\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_3 - e\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_5\end{aligned}$$

$$\begin{aligned}\left(\mathbf{k}_5 - \mathbf{k}_O\right)\cdot\mathbf{\Omega}\times\mathbf{r}_5 &= \mathbf{\Omega}\cdot\mathbf{r}_5\times\left(\mathbf{k}_5 - \mathbf{k}_O\right) \\ &= \mathbf{\Omega}\cdot\mathbf{r}_5\times\left(e\left(\mathbf{r}_5 - \mathbf{r}_4\right) - \mathbf{k}_O\right) \\ &= e\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_5 - \mathbf{\Omega}\cdot\mathbf{r}_5\times\mathbf{k}_O\end{aligned}$$

$$\left(\mathbf{k}_4 - \mathbf{k}_5\right)\cdot\mathbf{\Omega}^{\text{periscope}}\times\mathbf{r}_{34}^{\text{periscope}} =$$

$$\begin{aligned}&= \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{34}^{\text{periscope}}\times\left(\mathbf{k}_4 - \mathbf{k}_5\right) \\ &= \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{34}^{\text{periscope}}\times\left(d\mathbf{r}_{34}^{\text{periscope}} - \left(e\mathbf{r}_{35}^{\text{periscope}} - e\mathbf{r}_{34}^{\text{periscope}}\right)\right) \\ &= -e\mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{34}^{\text{periscope}}\times\mathbf{r}_{35}^{\text{periscope}}\end{aligned}$$

$$\left(\mathbf{k}_5 - \mathbf{k}_O\right)\cdot\mathbf{\Omega}^{\text{periscope}}\times\mathbf{r}_{35}^{\text{periscope}} =$$

$$\begin{aligned}&= \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\left(\mathbf{k}_5 - \mathbf{k}_O\right) \\ &= \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\left(e\mathbf{r}_{35}^{\text{periscope}} - e\mathbf{r}_{34}^{\text{periscope}}\right) - \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{k}_O \\ &= -e\mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{r}_{34}^{\text{periscope}} - \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{k}_O\end{aligned}$$

Adding the two equations (26) we have

$$\begin{aligned}\omega_4 - \omega_O &= -d\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_3 - e\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_5 + e\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_5 - \mathbf{\Omega}\cdot\mathbf{r}_5\times\mathbf{k}_O \\ &\quad - e\mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{34}^{\text{periscope}}\times\mathbf{r}_{35}^{\text{periscope}} - e\mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{r}_{34}^{\text{periscope}} \\ &\quad - \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{k}_O \\ &= -d\mathbf{\Omega}\cdot\mathbf{r}_4\times\mathbf{r}_3 - \mathbf{\Omega}\cdot\mathbf{r}_5\times\mathbf{k}_O - \mathbf{\Omega}^{\text{periscope}}\cdot\mathbf{r}_{35}^{\text{periscope}}\times\mathbf{k}_O\end{aligned} \tag{28}$$

5.4 Doppler Shift Resulting from a Sequence of Mirrors within both the Lidar's Sea Container and its Periscope

Add (22) and (28), and use (27) to obtain

$$\begin{aligned}\omega_1 - \omega_4 + \omega_4 - \omega_O &= -\mathbf{k}_4 \cdot \mathbf{\Omega} \times \mathbf{r}_3 - d\mathbf{\Omega} \cdot \mathbf{r}_4 \times \mathbf{r}_3 - \mathbf{\Omega} \cdot \mathbf{r}_5 \times \mathbf{k}_O \\ &\quad - \mathbf{\Omega}^{\text{periscope}} \cdot \mathbf{r}_{35}^{\text{periscope}} \times \mathbf{k}_O \\ \omega_1 - \omega_O &= -d\mathbf{\Omega} \cdot \mathbf{r}_3 \times \mathbf{r}_4 - d\mathbf{\Omega} \cdot \mathbf{r}_4 \times \mathbf{r}_3 - \mathbf{\Omega} \cdot \mathbf{r}_5 \times \mathbf{k}_O \\ &\quad - \mathbf{\Omega}^{\text{periscope}} \cdot \mathbf{r}_{35}^{\text{periscope}} \times \mathbf{k}_O \\ &= -\mathbf{\Omega} \cdot \mathbf{r}_5 \times \mathbf{k}_O - \mathbf{\Omega}^{\text{periscope}} \cdot \mathbf{r}_{35}^{\text{periscope}} \times \mathbf{k}_O \\ &= -\mathbf{k}_O \cdot \left[\mathbf{\Omega} \times \mathbf{r}_5 + \mathbf{\Omega}^{\text{periscope}} \times \mathbf{r}_{35}^{\text{periscope}}\right]\end{aligned}$$

Recall that ω_1 is the frequency at the beam splitter, ω_O is the frequency of the laser light that exits into the atmosphere and $\mathbf{k}_O$ is its wave vector, $\mathbf{r}_5$ is the position vector from the beam splitter to the scanning mirror on the periscope, and $\mathbf{r}_{35}$ is the position vector from the first mirror in the periscope to the scanning mirror. Those position vectors are all measured to the centers of rotation of the mirrors. As the periscope rotates, both $\mathbf{r}_5$ and $\mathbf{r}_{35}$ change with time. We can write $\mathbf{r}_5 = \mathbf{r}_3 + \mathbf{r}_{35}^{\text{periscope}}$; eliminating $\mathbf{r}_5$ is convenient because $\mathbf{r}_3$ is the position vector from the beam splitter to the first mirror in the periscope; as such, $\mathbf{r}_3$ does not change as the periscope rotates.

Let us choose more descriptive notation for the position vectors. Let $\mathbf{r}_3 = \mathbf{r}_{\text{splitter to periscope}}$ and $\mathbf{r}_{35}^{\text{periscope}} = \mathbf{r}_{\text{periscope to scanner}}$. We can write

$$\omega_1 - \omega_O = -\mathbf{k}_O \cdot \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \tag{29}$$

Note that $\left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right)$ is the total angular rate of the periscope. As a check of (29) we can consider the case when the periscope is not rotating such that $\mathbf{\Omega}^{\text{periscope}} = 0$, then $\left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] = \mathbf{\Omega} \times \left(\mathbf{r}_{\text{splitter to periscope}} + \mathbf{r}_{\text{periscope to scanner}}\right) = \mathbf{\Omega} \times \mathbf{r}_{\text{splitter to scanner}}$; this is the correct result.

5.5 Relationship of Measured Doppler Frequency to Atmospheric and Lidar Motions

Light scattered from the atmosphere enters the lidar and traverses the reverse path from the scanning mirror to the beam splitter. Assume that the motion of the lidar has changed insignificantly during the time that a given lidar pulse has traveled from the scanning mirror and back to that mirror. Then, one can analyze this situation by reversing the direction of the wave vectors in the above analysis. Let $\mathbf{k}_I$ be the wave vector of light returning from atmospheric scattering which is incident on the center of rotation of the scanning mirror. Let the associated frequency be denoted ω_I. Let ω_B denote the frequency of the returned light at the beam splitter. Similar to (29) we have

$$\omega_I - \omega_B = \mathbf{k}_I \cdot \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \tag{30}$$

Adding (29) and (30), the measured Doppler shift $\omega_1 - \omega_B$ is

$$\omega_1 - \omega_B = (\omega_O - \omega_I) + (\mathbf{k}_I - \mathbf{k}_O) \cdot \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \tag{31}$$

The difference $\omega_O - \omega_I$ is the Doppler shift of the outgoing versus incoming light in the atmosphere. Recall that the above derivation used the inertial reference frame that coincides with the position of the beam splitter at time t, and has the same velocity as the beam splitter at that time t. Let us call that inertial reference frame the splitter's frame. Thus, $\omega_O - \omega_I$ is the Doppler shift in the splitter's frame. Light from the beam splitter goes both to the mixer and toward the atmosphere; $\omega_O - \omega_I$ gives the radial component of atmospheric velocity in the splitter's frame. Now we apply (17) to the scattering from the atmospheric aerosol as observed in the splitter's frame; we obtain

$$\omega_O - \omega_I = (\mathbf{k}_O - \mathbf{k}_I) \cdot \mathbf{V}_{\text{atmos}}^{\text{splitter frame}} \tag{32}$$

The atmospheric velocity observed in the splitter's frame is related to the atmospheric velocity observed in the Earth-fixed reference frame by

$$\mathbf{V}_{\text{atmos}}^{\text{splitter frame}} = \mathbf{V}_{\text{atmos}}^{E} - \mathbf{V}_{\text{splitter}}^{E} \tag{33}$$

where $\mathbf{V}^E_{\text{atmos}}$ is the desired velocity of the atmosphere in the Earth-fixed reference frame, and $\mathbf{V}^E_{\text{splitter}}$ is the velocity of the beam splitter as observed from the Earth-fixed reference frame.

Let $\mathbf{V}^E_{\text{scanner}}$ be the velocity of the scanning mirror as observed from the Earth-fixed reference frame. Analogous to (3) we have

$$\mathbf{V}^E_{\text{scanner}} = \mathbf{V}^E_{\text{splitter}} + \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right]^E \tag{34}$$

Of course, (34) is nothing more than recognition that $\left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right]$ is the velocity of the scanning mirror relative to the beam splitter, which in the Earth's coordinate system is $\left[\mathbf{V}^E_{\text{scanner}} - \mathbf{V}^E_{\text{splitter}}\right]$.

Substituting (33) into (32) and the resultant $\omega_O - \omega_I$ into (31) gives the measured Doppler frequency shift $\omega_1 - \omega_B$ as

$$\omega_1 - \omega_B = (\mathbf{k}_O - \mathbf{k}_I) \cdot \left(\mathbf{V}^E_{\text{atmos}} - \mathbf{V}^E_{\text{splitter}}\right) - (\mathbf{k}_O - \mathbf{k}_I) \cdot \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \tag{35}$$

If we also substitute (34) we have

$$\omega_1 - \omega_B = (\mathbf{k}_O - \mathbf{k}_I) \cdot \left(\mathbf{V}^E_{\text{atmos}} - \mathbf{V}^E_{\text{scanner}}\right) \tag{36}$$

Now (36) is really a short notation for (35) because to accurately determine $\mathbf{V}^E_{\text{scanner}}$ one must evaluate (34).

5.6 Approximation for $(\mathbf{k}_O - \mathbf{k}_I)$

The quantity $(\mathbf{k}_O - \mathbf{k}_I)$ is the difference of the outgoing and incoming wave vectors of the lidar's light as observed in the splitter's reference frame. Let us simplify this quantity by assuming a frequency of the lidar's light in the splitter's reference frame denoted by ω_{lidar}, and approximating $\mathbf{k}_O \simeq \widehat{\mathbf{p}}_O\, (\omega_{\text{lidar}}/c)$, where $\widehat{\mathbf{p}}_O$ is a unit vector in the direction of $\mathbf{k}_O$ at the moment that the laser pulse was emitted. Similarly approximate $-\mathbf{k}_I \simeq \widehat{\mathbf{p}}_I\, (\omega_{\text{lidar}}/c)$, where $\widehat{\mathbf{p}}_I$ is a unit vector in the direction of $-\mathbf{k}_I$ at the moment the laser pulse was received. Then,

$$(\mathbf{k}_O - \mathbf{k}_I) \simeq 2\frac{\omega_{\text{lidar}}}{c}\mathbf{p}$$

where we define

$$\mathbf{p} \equiv (\widehat{\mathbf{p}}_O + \widehat{\mathbf{p}}_I)/2$$

Let θ_{OI} be the small angle through which the scanning mirror rotates between emission of the light beam and reception of that beam's scattered light. Since $|\mathbf{p}|^2 = \left(|\widehat{\mathbf{p}}_O|^2 + |\widehat{\mathbf{p}}_I|^2 + 2\widehat{\mathbf{p}}_O \cdot \widehat{\mathbf{p}}_I\right)/4 = (1 + 1 + 2\cos\theta_{OI})/4 \simeq \left(2 + 2\left(1 - \frac{1}{2}(\theta_{OI})^2\right)\right)/4 \simeq 1 - \frac{1}{4}(\theta_{OI})^2$, we have $|\mathbf{p}| = 1 - \frac{1}{8}(\theta_{OI})^2$. If one neglects the change of orientation of the scanning mirror during the time the light is in the atmosphere, then $\mathbf{k}_I$ is antiparallel to $\mathbf{k}_O$ such that $\widehat{\mathbf{p}}_O \simeq \widehat{\mathbf{p}}_I$ and $|\mathbf{p}| = 1$.

5.6 Radial Component of Atmospheric Velocity

Given these approximations, (35) and (36) give the lidar's radial component of atmospheric velocity in terms of the measured Doppler frequency shift $\omega_1 - \omega_B$

$$\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}} = (\omega_1 - \omega_B)/\left(2\frac{\omega_{\text{lidar}}}{c}\right) + \mathbf{p} \cdot \mathbf{V}^E_{\text{splitter}} + \mathbf{p} \cdot \left[\mathbf{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\mathbf{\Omega} + \mathbf{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \tag{37}$$

$$\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}} = (\omega_1 - \omega_B)/\left(2\frac{\omega_{\text{lidar}}}{c}\right) + \mathbf{p} \cdot \mathbf{V}^E_{\text{scanner}} \tag{38}$$

5.8 Motion Correction of the Atmospheric Velocity Measured by the Lidar

To calculate the radial component of atmospheric velocity, $\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}}$, from (37) or (38) we must determine either $\mathbf{V}^E_{\text{splitter}}$ or $\mathbf{V}^E_{\text{scanner}}$ from either the ship's or the lidar's motion compensation systems. To do so we must apply (3).

The ship's POS MV system gives the velocity $\mathbf{V}^E$ of a point on the ship and the angular rate $\boldsymbol{\Omega}$. The ship's system was surveyed to give that information recorded as "Sensor 1" at the red accelerometer box in the lidar's sea container. The lidar's motion compensation system gives the same information except that the velocity $\mathbf{V}^E$ is that at the GPS master antenna (according to the manufacturer). Because that antenna is atop the lidar sea container and diagonally opposite the lidar's periscope, and because the center of the sea container was above the ship's roll axis, calculation of the motion of the scanning mirror is essential. Whether using the lidar's or ship's motion measurements, let $\mathbf{r}_{\text{point to splitter}}$ denote the position vector from the point where the Earth frame velocity is $\mathbf{V}^E$ to the beam splitter. The Earth-frame velocity of the beam splitter, denoted by $\mathbf{V}^E_{\text{splitter}}$, and that of the scanner, are then determined from (3), i.e., from

$$\mathbf{V}^E_{\text{splitter}} = \mathbf{V}^E + (\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}})^E \tag{39}$$

One must be careful when using this equation because $\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}}$ is calculated in either the ship's coordinate system or the lidar's coordinate system and must be expressed in the Earth's coordinate system before it is added to $\mathbf{V}^E$ (see Sec. 2.2 for how to calculate $(\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}})^E$). By definition, $\mathbf{V}^E$ has its components determined in the Earth's coordinate system. Calculating $(\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}})^E$ from $\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}}$ requires the coordinate transformation matrix $\mathbf{Q}$ described above, but account must be taken that the lidar's coordinate system is rotated relative to the ship's coordinate system.

One can obtain $\mathbf{V}^E_{\text{splitter}}$ from (39) and substitute $\mathbf{V}^E_{\text{splitter}}$ into (37) to obtain $\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}}$. Alternatively, one can substitute (39) into (37) to obtain

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}} &= (\omega_1 - \omega_B) \,/\, \left(2\frac{\omega_{\text{lidar}}}{c}\right) + \mathbf{p} \cdot \mathbf{V}^E \\
&\quad + \mathbf{p} \cdot \left[\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to splitter}} + \boldsymbol{\Omega} \times \mathbf{r}_{\text{splitter to periscope}} + \left(\boldsymbol{\Omega} + \boldsymbol{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right] \\
\mathbf{p} \cdot \mathbf{V}^E_{\text{atmos}} &= (\omega_1 - \omega_B) \,/\, \left(2\frac{\omega_{\text{lidar}}}{c}\right) + \mathbf{p} \cdot \mathbf{V}^E + \mathbf{p} \cdot \left[\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to periscope}} + \left(\boldsymbol{\Omega} + \boldsymbol{\Omega}^{\text{periscope}}\right) \times \mathbf{r}_{\text{periscope to scanner}}\right]
\end{aligned}$$

where $\mathbf{r}_{\text{point to periscope}} \equiv \mathbf{r}_{\text{point to splitter}} + \mathbf{r}_{\text{splitter to periscope}}$. Of course, if $\boldsymbol{\Omega}^{\text{periscope}} = 0$ then the right most term simplifies to

$$\left[\boldsymbol{\Omega} \times \mathbf{r}_{\text{point to periscope}} + \boldsymbol{\Omega} \times \mathbf{r}_{\text{periscope to scanner}}\right] = \boldsymbol{\Omega} \times \mathbf{r}_{\text{point to scanner}}$$

6. COMPARISON OF MOTION CORRECTIONS USING THE SHIP'S DATA VERSUS THE LIDAR'S DATA

There are several reasons to compare correcting the various instruments's data using the ship's motion detection data with the same correction using the Lidar's motion detection data. The manufacturer of the lidar's system referred to the lidar's master GPS as the point where the velocity is reported. The accuracy of that claim can be check by comparison with the ship's motion detection data. There were many drop outs of the lidar's motion detection when the GPS lost signal. The ship's data is more continuous. It is important to assess the accuracy of the lidar's motion detection as compared to the ship's motion detection data. In this section methods for quantifying such comparisons are given. First, more specific notation is needed which includes both the coordinate system and the measuring system for each type of data. That specific notation is described in he next section. Subsequent sections give regression algorithms that determine quantities of interest and examples of uses for the results.

6.1 Notation for Quantities Measured by the Motion Detection Systems

A superscript L will denote a vector's components in the lidar's motion detection coordinate system. A superscript S will denote a vector's components in the ship's motion detection coordinate system. A superscript E will denote a vector's components in the Earth's coordinate system, that is, north, east, down in that order. The phrase "in the coordinate system" means that the vector's components are obtained by projection of the vector along the coordinate axes, that is, by inner product of the vector with the unit vectors that are aligned with the positive direction of each axis. The lidar's and ship's coordinate systems have their origins spatially displaced from one another by a fixed separation vector, and they are in fixed orientation relative to one another. The ship's coordinate system is forward (x), starboard (y) and down (z). Both coordinate systems are assumed to be right handed in the order (x, y, z) which are the names of the axes; (x, y, z) corresponds to and can be replaced by numerical indices $(1, 2, 3)$. Rotation about the x axis in the right-handed sense is called 'roll' ϕ; rotation about the y axis in the right-handed sense is called 'pitch' θ; rotation about the z axis in the right-handed sense is called 'heading' or 'yaw'.

Both lidar and ship coordinate systems are translating relative to the Earth's coordinate system which is fixed relative to the Earth's lithosphere. A given velocity $\mathbf{v}$ is denoted by $\mathbf{v}^S$ when its components are in the ship's coordinate system and by $\mathbf{v}^L$ when its components are in the lidar's coordinate system and by $\mathbf{v}^E$ when its components are in the Earth's coordinate system. Both the lidar's and ship's coordinate systems are rotating relative to a coordinate system that is fixed relative to the Earth's lithosphere by angular rate denoted by $\boldsymbol{\Omega}^S$ when its components are in the ship's coordinate system and by $\boldsymbol{\Omega}^L$ when its components are in the lidar's coordinate system. $\boldsymbol{\Omega}^L$ and $\boldsymbol{\Omega}^S$ are the same vector because the ship is a rigid body.

A subscript L denotes a quantity measured by the lidar's motion detection system, and a subscript S denotes a quantity measured by the lidar's motion detection system. No subscript appears on quantities calculated from the measured quantities. Although $\boldsymbol{\Omega}^L$ and $\boldsymbol{\Omega}^S$ are the same vector, because of measurement errors $\boldsymbol{\Omega}_S^S$ and $\boldsymbol{\Omega}_L^L$ differ; that difference is a function of time because of random measurement errors. The analogous statement is not true of $\mathbf{v}_S^E$ and $\mathbf{v}_L^E$ because the velocity at the lidar differs from the velocity at the origin of the coordinate system of the ship's motion detection system.

6.2 Measured Quantities and Units

If the data is in units other than are stated here, then the data should be changed to the units here. In particular, angles and angular rates are probably in degrees and degrees per second, respectively.

Ω_L^L is the vector of angular rates in radians per second measured by the lidar's motion detection system.

Ω_S^S is the vector of angular rates in radians per second measured by the ship's motion detection system.

v_L^E is the velocity vector in meters per second measured by the lidar's motion detection system with its components in the Earth's coordinate system (north,east,down).

v_S^E is the velocity vector in meters per second measured by the ship's motion detection system with its components in the Earth's coordinate system (north,east,down).

$(\phi_L, \theta_L, \psi_L)$ are the Euler angles from the lidar's motion detection system determined from integration of the angular rates Ω_L^L.

$(\phi_S, \theta_S, \psi_S)$ are the Euler angles from the ship's motion detection system determined from integration of the angular rates Ω_S^S.

t_L is the sequence of times in seconds at which Ω_L^L and $\mathbf{v}_L^E$ are measured.

t_S is the sequence of times in seconds at which Ω_S^S and $\mathbf{v}_S^E$ are measured.

6.3 Quantities to be Determined by Regression: $\mathbf{r}^S(SL)$ and $\mathbf{R}$

$\mathbf{r}^S(SL)$ is the position vector in meters that points from the location on the ship where the velocity is $\mathbf{v}_S^E$ to the location at the lidar where the velocity is $\mathbf{v}_L^E$. Since it has superscript S, the components of $\mathbf{r}^S(SL)$ are in the ship's motion detection coordinate system. $\mathbf{R}$ is the 3 by 3 coordinate transformation matrix that transforms a vector from the lidar's coordinate system to the ship's coordinate system. $\mathbf{R}$ is expressed in terms of the Euler angles of that coordinate transformation by

$$\mathbf{R} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\theta\cos\psi\sin\phi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi & \cos\theta\cos\phi \end{bmatrix} \tag{40}$$

The quantities to be determined by regression are the 3 components of $\mathbf{r}^S(SL)$ and the 3 Euler angles (ϕ, θ, ψ). Of course, $\mathbf{R}$ is also the matrix of 9 direction cosines formed by the inner products of the unit vectors aligned along the positive axes of the ship's coordinate system with the unit vectors aligned along the positive axes of the lidar's coordinate system.

The time sequences must coincide, but the sample times are not necessarily the same, that is, $t_L \neq t_S$. Assume that the data rates are unequal. If the lidar has the faster time series, then linearly interpolate the values of Ω_L^L and $\mathbf{v}_L^E$ to the slower time sequence t_S. If the ship has the faster time series, then linearly interpolate the values of Ω_S^S and $\mathbf{v}_S^E$ to the times t_L. If the data are synchronized, then no interpolation is needed.

6.4 Equations Used for the Regression

First, consider the equations for the rotation matrix $\mathbf{R}$. At every position on the ship, the angular rates are the same because the ship is a rigid body. Therefore, by definition of $\mathbf{R}$, and neglecting the measurement errors,

$$\boldsymbol{\Omega}_S^S = \mathbf{R}\boldsymbol{\Omega}_L^L \tag{41}$$

This constitutes 3 nonlinear transcendental equations for the 3 unknown Euler angles. It can be solved at each time using a Newton-Rapheson subroutine. The solution will vary with time because of random errors in Ω_S^S and Ω_L^L. It is easier to use all of the time series in a regression subroutine to obtain the Euler angles and their random errors. It may be yet easier to ignore the Euler angle formulation and obtain all 9 components of $\mathbf{R}$ by regression.

At this point, we know $\mathbf{R}$. A reality check can be performed. Further, $\mathbf{R}$ should be an orthogonal transformation such that its transpose is its inverse. Therefore, test that

$$\mathbf{R}^T = \mathbf{R}^{-1} \tag{42}$$

Let I be the identity matrix. The test of (42) is that the 9 elements of the matrix

$$\mathbf{R}\mathbf{R}^T - I \tag{43}$$

should be small compared to unity.

Another way to get $\mathbf{R}$ is to use the Euler angles $(\phi_L, \theta_L, \psi_L)$ and $(\phi_S, \theta_S, \psi_S)$ which are functions of time. By definition, any vector $\mathbf{U}$ has its Earth, lidar, and ship's components related by

$$\mathbf{R}(\phi_L, \theta_L, \psi_L)\,\mathbf{U}^E = \mathbf{U}^L$$
$$\mathbf{R}(\phi_S, \theta_S, \psi_S)\,\mathbf{U}^E = \mathbf{U}^S$$

where $\mathbf{R}(\phi_L, \theta_L, \psi_L)$ is (40) with (ϕ, θ, ψ) replaced by $(\phi_L, \theta_L, \psi_L)$ and $\mathbf{R}(\phi_S, \theta_S, \psi_S)$ is (40) with (ϕ, θ, ψ) replaced by $(\phi_S, \theta_S, \psi_S)$. Thus,

$$\mathbf{U}^E = \mathbf{R}^T(\phi_L, \theta_L, \psi_L)\,\mathbf{U}^L$$
$$\mathbf{U}^E = \mathbf{R}^T(\phi_S, \theta_S, \psi_S)\,\mathbf{U}^S$$

Eliminating $\mathbf{U}^E$ gives $\mathbf{R}^T(\phi_L, \theta_L, \psi_L)\,\mathbf{U}^L = \mathbf{R}^T(\phi_S, \theta_S, \psi_S)\,\mathbf{U}^S$, thus,

$$\mathbf{U}^S = \mathbf{R}(\phi_S, \theta_S, \psi_S)\,\mathbf{R}^T(\phi_L, \theta_L, \psi_L)\,\mathbf{U}^L$$

from which the definition of $\mathbf{R}$ in (40) gives

$$\mathbf{R} = \mathbf{R}(\phi_S, \theta_S, \psi_S)\,\mathbf{R}^T(\phi_L, \theta_L, \psi_L)$$

$\mathbf{R}$ should be independent of time whereas $\mathbf{R}(\phi_L, \theta_L, \psi_L)$ and $\mathbf{R}(\phi_S, \theta_S, \psi_S)$ vary with time. $\mathbf{R}$ will vary with time because of random errors in $(\phi_L, \theta_L, \psi_L)$ and $(\phi_S, \theta_S, \psi_S)$.

Now, consider the velocity at a given point on the ship. Consider a position vector $\mathbf{r}^S$ that points from the ship's coordinate origin where the velocity is $\mathbf{v}_S^E$, as measured by the ship's motion detection system, to any other location. Let $\mathbf{V}^E(\mathbf{r})$ be the velocity of that location on the ship. Since $\mathbf{V}^E$ has superscript E, $\mathbf{V}^E$ is expressed in the Earth's coordinate system. The velocity $\mathbf{V}^E\left(\mathbf{r}^S\right)$ as determined by the ship's motion detection system is

$$\mathbf{V}^E\left(\mathbf{r}^S\right) = \mathbf{v}_S^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S\right)^E \tag{44}$$

How to calculate $\left(\mathbf{\Omega}^S \times \mathbf{r}^S\right)^E$ is given in Section 2.2. In particular, the velocity at the lidar's motion detection system, as determined by the ship's motion detection system, is

$$\mathbf{V}^E\left(\mathbf{r}^S(SL)\right) = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E \tag{45}$$

where the argument (SL) of $\mathbf{r}^S(SL)$ denotes ship origin to lidar origin. Recall that the velocity of that point as measured by the lidar's motion detection system is $\mathbf{v}_L^E$. Equating the ship's and lidar's measured velocities, i.e., $\mathbf{V}^E\left(\mathbf{r}^S(SL)\right) = \mathbf{v}_L^E$, gives an equation for $\mathbf{r}^S(SL)$, namely,

$$\mathbf{v}_L^E = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E \tag{46}$$

Note that we have neglected the measurement errors in $\mathbf{v}_L^E$ and $\mathbf{v}_S^E$; those errors produce error in $\mathbf{r}^S(SL)$, as do errors in $\mathbf{R}$ and Ω_S. Equation (46) constitutes 3 linear algebraic equations for the 3 unknown components of $\mathbf{r}^S(SL)$. It can be solved at each time using either a matrix inversion or an analytically inverted 3 by 3 matrix. The solution will vary with time because of random errors in $\mathbf{v}_S^E$ and $\mathbf{v}_L^E$. It is easier to use all of the time series in a regression subroutine to obtain the 3 components of $\mathbf{r}^S(SL)$ and their random errors.

At this point we know $\mathbf{r}^S(SL)$. A reality check can be performed.

7. USES FOR THE RESULTS IN SECTION 6

7.1 Correct the Lidar's Data Using the Lidar's Motion Detection System

The lidar's motion correction system consists of 4 GPS antennas positioned a few feet beyond the corners of the roof of the lidar's sea container and an accelerometer box attached to the inside ceiling of the sea container. That system reports the velocity in the Earth's coordinate system for some point. It was not clear where that point was at the beginning of the cruise of the Seward Johnson. Later, the manufacturer reported that the velocity was reported for the position of the master GPS antenna; the accuracy of the manufacturer's statement is unknown, but it can be tested with the present formulation. It is the motion of the scanning mirror that contaminates the Doppler measurement **(see section)**. Unfortunately the lidar's master GPS antenna is diagonally opposite the scanning mirror. Therefore, consider the determination of the velocity of the scanning mirror using the lidar's data.

Let $\mathbf{r}^L$ be the position vector in the lidar's coordinate system that points from the lidar's master GPS antenna to the scanning mirror. Because of the rotation of the scanning mirror's periscope, $\mathbf{r}^L$ is a function of time. The variant of (46) that gives the velocity of the scanning mirror, i.e., $\mathbf{v}^E(\mathbf{r}_{\text{scanning}})$, from the velocity reportedly at the master GPS antenna, i.e., $\mathbf{v}_L^E$, is

$$\mathbf{v}^E(\mathbf{r}_{\text{scanning}}) = \mathbf{v}_L^E + \left(\mathbf{\Omega}_L^L \times \mathbf{r}^L\right)^E \tag{47}$$

7.2 Correct the Lidar's Data Using the Ship's Motion Detection System

Multiply (41) by $\mathbf{R}^{-1}$. to obtain

$$\mathbf{\Omega}^L = \mathbf{R}^{-1}\mathbf{\Omega}_S^S \tag{48}$$

Since we now know $\mathbf{r}^S(SL)$ we use it in (46) to give $\mathbf{v}^E\left(\mathbf{r}^S(SL)\right)$ as a substitute for $\mathbf{v}_L^E$ from

$$\mathbf{v}^E\left(\mathbf{r}^S(SL)\right) = \mathbf{v}_S^E + \left(\mathbf{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E \tag{49}$$

We thereby obtain the quantities needed by the lidar from the ship's motion detection, namely $\mathbf{v}^E$ at the lidar from (49) and Ω^L from (48).

7.3 Correct NOAA/K Doppler Velocities Using the Ship's Motion Detection System

Let $\mathbf{r}^S(SK)$ denote the position vector in meters that points from the location on the ship where the velocity is measured to be $\mathbf{v}_S^E$ to the location of the NOAA/K antenna. $\mathbf{r}^S(SK)$ must be measured. Since $\mathbf{r}^S(SK)$ has superscript S, the components of $\mathbf{r}^S(SK)$ are in the ship's motion detection coordinate system. From (44) the velocity vector at the NOAA/K antenna is

$$\mathbf{V}^E\left(\mathbf{r}^S(SK)\right) = \mathbf{v}_S^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(SK)\right)^E \tag{50}$$

Ω_S^S is recorded to determine the pointing direction of the NOAA/K antenna. Of course, Ω_L^L is also recorded such that Ω_L^L can be used to determine the pointing direction of the NOAA/K antenna. It is also possible in the future to use Ω_S to correct the pointing of the NOAA/K antenna. Aboard the Seward Johnson, the NOAA/K antenna was surveyed to become the ship's coordinate origin (see the appendix). Hence, this option was obviated because $r^S(SK) = 0$.

7.4 Correct NOAA/K Doppler Velocities Using the Lidar's Motion Detection System

As above, $\mathbf{r}^S(SK)$ denotes the position vector in meters that points from the location on the ship where the velocity is $\mathbf{v}_S^E$ to the location of the NOAA/K antenna. Solve (46)$\mathbf{v}_L^E = \mathbf{v}_S^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E$ for $\mathbf{v}_S^E$ and substitute it into (50)$\mathbf{V}^E\left(\mathbf{r}^S(SK)\right) = \mathbf{v}_S^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(SK)\right)^E$. From (44)$\mathbf{V}^E\left(\mathbf{r}^S\right) = \mathbf{v}_S^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S\right)^E$ The velocity vector at the NOAA/K antenna is

$$\begin{aligned}
\mathbf{V}^E\left(\mathbf{r}^S(SK)\right) &= \mathbf{v}_L^E - \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(SL)\right)^E + \left(\boldsymbol{\Omega}_S^S \times \mathbf{r}^S(SK)\right)^E \\
\mathbf{V}^E\left(\mathbf{r}^S(SK)\right) &= \mathbf{v}_L^E + \left(\boldsymbol{\Omega}_S^S \times \left[\mathbf{r}^S(SK) - \mathbf{r}^S(SL)\right]\right)^E
\end{aligned} \tag{51}$$

Finally, replace the ship's measurement Ω_S^S with the angular rate derived from the lidar's measurement according to (41), i.e., $\Omega^S = \mathbf{R}\Omega_L^L$, to obtain

$$\mathbf{V}^E\left(\mathbf{r}^S(SK)\right) = \mathbf{v}_L^E + \left(\left[\mathbf{R}\boldsymbol{\Omega}_L^L\right] \times \left[\mathbf{r}^S(SK) - \mathbf{r}^S(SL)\right]\right)^E \tag{52}$$

Note that $\mathbf{r}^S(SK) - \mathbf{r}^S(SL)$ is the position vector in meters that points from the location where the lidar measures the velocity $\mathbf{v}_L^E$ to the location of the NOAA/K antenna. The components of $\mathbf{r}^S(SK) - \mathbf{r}^S(SL)$ are in the ship's coordinate system. Note that this contains the lidar's measured velocity and angular rate rotated to the ship's motion detection coordinate system and contains the cross product with $\mathbf{r}^S(SK) - \mathbf{r}^S(SL)$ performed in the ship's coordinate system. Although Ω_L^L is recorded to determine the pointing direction of the NOAA/K antenna, $\Omega^S = \mathbf{R}\Omega_L^L$ is calculated to correct the pointing. The reason is that NOAA/K antenna pointing is measured in the ship's coordinate system.

7.5 Correct the U. Miami's X- and W-Band Doppler Velocities Using Either the Lidar's or Ship's Motion Detection System

The method is the same as above for NOAA/K. Let $\mathbf{r}^S(SX)$ and $\mathbf{r}^S(SW)$ denote the position vector in meters that points from the location on the ship where the velocity is measured to be $\mathbf{v}_S^E$ to the location of the X-band and W-band antennas, respectively. $\mathbf{r}^S(SX)$ and $\mathbf{r}^S(SW)$ must be measured. $\mathbf{r}^S(SX)$ and $\mathbf{r}^S(SW)$ are in the ship's motion detection coordinate system. Replace $\mathbf{r}^S(SK)$ in (50) or (52) with either $\mathbf{r}^S(SX)$ or $\mathbf{r}^S(SW)$ to obtain the respective correction.

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12. REFERENCES

Corcoran, R., and R. Pronk, 2003. POS MV Model 320 V30 Ethernet & SCSIICD, (available from the manufacturer)

Edson, J. B., A. A. Hinton, K. E. Prada, J. E. Hare, and C. W Fairall, 1998. Direct Covariance Flux Estimates from Mobile Platforms at Sea. *J. Atmos. Ocean. Tech.* 15:547-562.

Schulz, E. W., B. G. Sanderson, and E. F. Bradley, 2005. Motion Correction for Shipborne Turbulence Sensors. *J. Atmos. Ocean. Tech.* 22:55-69.

Appendix A: Surveyed Coordinates to Use in the Ship's Inertial Measurement Unit

The POS MV consists of an Inertial Measurement Unit (IMU), a Primary GPS unit, and the ship's GPS2 unit. The Primary GPS unit has its antenna above the aft 01 deck. The ship's GPS2 unit, also designated POS MV GPS2, has its antenna above the aft 01 deck and is connected to a Ray Marine GPS receiver box in the Dry Lab. The IMU is the accelerometer's box on the floor of the computer lab. The POS MV allows a point on the ship to be designated as the reference point and two other points designated as Sensor1 and Sensor2. For each of those three points, the POS MV unit outputs the velocities of those three points in the Earth's reference frame, the angular acceleration, and the orientation (Euler angles), as well as accuracy metrics for those quantities. Details of the POS MV data are given in another document.

Reference point **X** drilled into the deck outside the compressor room is referred to as point 'ref.' Other points on the ship are surveyed from that reference point. That reference point is about 4 feet = 48/39 m= 1.23 m above sea level during measurements. Recall that the ship's coordinate system is forward (x), starboard (y) and down (z) and that coordinates of a point are given as $\begin{bmatrix} x & y & z \end{bmatrix}$. The following are coordinates of position vectors from ref. to other points as measured in meters:

ref. to Primary GPS antenna $G = \begin{bmatrix} 0.0 & -3.784 & -6.144 \end{bmatrix}$
ref. to POS MV GPS2 $A = \begin{bmatrix} 9.440 & -6.379 & -3.974 \end{bmatrix}$
ref. to IMU is $I = \begin{bmatrix} 19.567 & -4.128 & -0.15 \end{bmatrix}$
ref. to Lidar is $L = \begin{bmatrix} 32.61 & 0.82 & -4.84 \end{bmatrix}$
ref. to NOAA/K radar is $K = \begin{bmatrix} 11.40 & 0.84 & -5.30 \end{bmatrix}$
ref. to roof W-band radar is $W = \begin{bmatrix} -9.19 & -2.88 & -2.88 \end{bmatrix}$
ref. to compensated W-band radar is $C = \begin{bmatrix} -6.22 & -1.18 & -1.42 \end{bmatrix}$
ref. to roof X-band radar is $R = \begin{bmatrix} -10.26 & -2.03 & -2.99 \end{bmatrix}$
ref. to center of rotation of the ship (estimated) is $C = \begin{bmatrix} 0.0 & -0.94 & 0.0 \end{bmatrix}$

The center of rotation of the ship is estimated to be at wall which is 0.94 m toward port side of the drilled **X** reference point. The position of the lidar is the red accelerometer box on the roof of the Lidar sea container. The position of any of the radars is an estimate of the phase center of the radar's antenna.

Usually, the POS MV system would report the ship's data for point ref. For our purposes it is more convenient to redefine the reference point as the NOAA/K radar. Let NOAA/K radar be the new reference point entered into the IMU. The the 'lever' arms to be entered into the POS MV computer data base for its GPS antennas, its new reference point 'ref', and Sensor1 and Sensor2 positions are:

NOAA/K to Primary GPS antenna: $G - K = \begin{bmatrix} -11.4 & -4.624 & -0.844 \end{bmatrix}$
NOAA/K to POS MV GPS2: $A - K = \begin{bmatrix} -1.96 & -7.219 & 1.326 \end{bmatrix}$
NOAA/K to IMU: $I - K = \begin{bmatrix} 8.167 & -4.968 & 5.15 \end{bmatrix}$
NOAA/K to Lidar: $L - K = \begin{bmatrix} 21.21 & -0.02 & 0.46 \end{bmatrix}$
NOAA/K to roof W-band radar: $W - K = \begin{bmatrix} -20.59 & -3.72 & 2.42 \end{bmatrix}$
NOAA/K to compensated W-band radar $C - K = \begin{bmatrix} -17.62 & -2.02 & 3.88 \end{bmatrix}$
NOAA/K to roof X -band radar: $R - K = \begin{bmatrix} -21.66 & -2.87 & 2.31 \end{bmatrix}$
NOAA/K to center of rotation of the ship $C - K = \begin{bmatrix} -11.4 & -1.78 & 5.3 \end{bmatrix}$

Sensor 1 has been assigned to the lidar accelerometer box. At 13:40 UTC on January 13, Sensor 2 was changed to the roof X-band. Prior to that time, Sensor 2 was the roof W-band radar (which ceased to function after January 13). The Univ. Miami's priority list is the following order from most to least important: roof W-band radar, compensated W-band radar, and roof X-band radar. The position of the Lidar accelerometer box center is recorded above for comparison with lidar motion data. The lidar's scanning mirror changes its position relative to the lidar accelerometer as its periscope rotates. The periscope center is displaced about 21 inches forward, 15 inches starboard, and the mirror center is 37 + 5 inches above the IMU box.

The instrumented tower was surveyed on January 28, 2005 which was after the above coordinates were recorded. The base of the tower is on the 01 deck 4.20 m above the water line. The sonic anemometer measurement volume is 10.1 m above the tower base. The center of the tower base was 1.55 m to port of the ship's center and 6.35 m forward from the rear edge of the Lidar sea container. The Lidar is 1.77 m starboard of the ship's center to within 2 cm. The Lidar's accelerometer box is its reference point. From the top of the Lidar sea container that accelerometer box is 0.13 m down. The top of the Lidar sea container is 2.64 m above the 01 deck. Relative to the Lidar reference point, the coordinates of the base of the tower are:

Tower base to Lidar $T-L = \left[\ 6.35-1.00 \ \ -1.55-1.77 \ \ 2.64-0.13\ \right] = \left[\ 5.35 \ \ -3.32 \ \ 2.51\ \right]$

Hence the coordinates of NOAA/K to the tower base are:

$T-K = [T-L]+[L-K] = \left[\ 5.35 \ \ -3.32 \ \ 2.51\ \right] + \left[\ 21.21 \ \ -0.02 \ \ 0.46\ \right] = \left[\ 26.56 \ \ -3.34 \ \ 2.97\ \right]$

Thus, NOAA/K to Tower base is $T-K = \left[\ 26.56 \ \ -3.34 \ \ 2.97\ \right]$

Hence, the coordinates of NOAA/K to the sonic anemometer's measurement volume are

$[T-K]+[S-T] = \left[\ 26.56 \ \ -3.34 \ \ 2.97\ \right] + \left[\ 0 \ \ 0 \ \ -10.1\ \right] = \left[\ 26.56 \ \ -3.34 \ \ -7.13\ \right]$

Thus, NOAA/K to Sonic is $S-K = \left[\ 26.56 \ \ -3.34 \ \ -7.13\ \right]$

The reference point ref. to the tower base is

$[T-L]+L = \left[\ 5.35 \ \ -3.32 \ \ 2.51\ \right] + \left[\ 32.61 \ \ 0.82 \ \ -4.84\ \right] = \left[\ 37.96 \ \ -2.5 \ \ -2.33\ \right]$

The vertical coordinate of -2.33 m agrees exactly with the surveyed height of the 01 deck above the reference point. The reason for the agreement is that the POS MV's coordinate system has its axes toward bow and port parallel to the main deck. On the other hand, the trim of the ship determines the tilt of the ship relative to sea level. Recall from above that the reference point ref is 1.23 m above sea level. From the above, one expects the tower base to be $(2.33+1.23)$ m$=$ 3.56 m above sea level if the ship was trimmed level to sea level. The measurement of the base of the tower on the 01 deck to the water line was 4.20 m, as noted above. We surmise that the decks rise toward the bow by $4.20-3.56$ m $=0.64$ m relative to the reference point. The corresponding angle is approximately $0.64/37.96 = 1.6860\times10^{-2}$ radians, which is one degree.